Necessary Conditions for Optimal Pulse Control

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Necessary conditions are derived for constrained variations that arise in the case of optimal pulse control. It is assumed that the pulse magnitudes and durations are fixed, and that the only variables subject to control are the pulse initiation times. It is shown that these conditions replace two standard necessary conditions for optimality. A missile application to demonstrate their validity and usefulness is considered.

Introduction

INTEREST in optimal pulse control arises from recent developments in pulsed solid rocket motor technology. These motors have varying degrees of in-flight propulsion control capability. The most promising candidates from a cost and reliability viewpoint are multiple-pulse motors with variable-pulse initiation times. Studies that address propulsion energy management or thrust magnitude control fall broadly into two categories. The first category contains approaches based on parameter optimization that can be solved using gradient algorithms.[1] The second category uses optimal control theory and model order reduction techniques to derive on-line thrust control algorithms.[2,3]

The purpose of this paper is to point out the fact that the standard necessary conditions, which are used as a basis for optimal control analysis, do not apply to the pulse control problem. New necessary conditions are derived based on the calculus of variations. The differences are due to the highly constrained variations associated with the pulse control problem. These constraints cannot be relaxed by the usual approach of adjoining them to the Hamiltonian. As a consequence, the Hamiltonian is not constant along an optimal solution, even for time-invariant formulations. Moreover, the first variation with respect to control does not vanish. Two new necessary conditions are developed for the case of scalar control, where the control appears linearly in the dynamics. The results of this paper an be viewed as an extention of earlier results on pulse control, where the pulses were modeled as impulses.[3] It is shown that the impulse control results are a subset of the more general class of problems treated here.

To demonstrate the validity and usefulness of the necessary conditions, a missile application is treated. When the dynamics are restricted to constant altitude a simple pulse triggering condition results in terms of the ratio of missile speed to aerodynamic drag. The result does not presuppose any analytical model for drag, and provides an exact solution to the multiple-pulse control problem.

Necessary Conditions for Optimality

In this section we first state the necessary conditions for optimality for an unconstrained optimal control problem, where the control is allowed to take on arbitrary variations. Next, necessary conditions are derived for constrained variations that arise in the pulse optimal control problem.

Unconstrained Optimal Control Problem

The classical optimal control problem is defined for a system described by a set of first-order differential equations of the form

$$\dot{x}=f(x,u,t), \qquad x(t_0)=x_0 \tag{1}$$

where x is an n-dimensional state vector, and u an m-dimensional control vector to be chosen to maximize a scalar cost functional of the form

$$J=\phi[x(t_f)]+\int_{t_0}^{t_f} L(x,u,t)\mathrm{d}t \tag{2}$$

subject to the scalar stopping condition $\psi[x(t_f)]=0$.

The formulation is unconstrained in the sense that $x(t)$ and $u(t)$ are allowed to take on any value over the interval $t\epsilon\ (t_0,t_f)$, and t_f is free. The reader is referred to Chap. 2 of Ref. 4 for a detailed account of the theoretical results for this problem formulation, including the effects of terminal constraints. A brief summary is given here for the unconstrained case, which was the problem of primary concern in this study.

Let $u^*(t)$ be the optimal control that maximizes Eq. (2), and let J^* be the resulting cost. Consider an arbitrary but small variation in control

$$u(t)=u^*(t)+\delta u(t) \tag{3}$$

that results in a small variation in cost defined by

$$J(u)=J^*+\delta J \tag{4}$$

Since $u^*(t)$ is the maximizing control, then for any arbitrary but small variaion $\delta u(t)$, it follows that

$$\delta J\leq 0 \tag{5}$$

In Ref. 4, it is shown that a first-order approximation for δJ is given by

$$\delta J=\int_{t_0}^{t_f} H_u(x,\lambda,u,t)\delta u(t)\mathrm{d}t\leq 0‡ \tag{6}$$

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‡Subscripts are used to denote a row vector of partial derivatives.

where H is a Hamiltonian defined by

$$H = \lambda^T f(x,u,t) + L(x,u,t) \tag{7}$$

and λ is an n-dimensional costate vector satisfying the differential equation

$$\dot{\lambda}^T = -H_x, \qquad \lambda^T(t_f) = \phi_x[x(t_f)] + \upsilon\psi_x[x(t_f)] \tag{8}$$

where υ is a free scalar parameter. Note that the only way the inequality in Eq. (6) can be satisfied for arbitrary $\delta u(t)$ is by insisting that

$$H_u(x,\lambda,u,t) = 0 \tag{9}$$

In summary, the necessary conditions for optimality are Eqs. (1), (8), and (9); that is, we must find a $u(t)$ that meets these conditions simultaneously. The problem is complicated by the fact that $x(t_f)$ is unknown, and the boundary condition on $\lambda(t)$ is given at the final time (also unknown) in terms of $x(t_f)$. This is referred to as a classical two-point boundary value problem. For free terminal time problems it can also be shown that

$$H[x(t_f),\lambda(t_f),u(t_f)t_f] = 0 \tag{10}$$

is also part of the necessary conditions, and is normally used to establish the final time. Note that when H is not an explicit function of t, i.e., when $f=f(x,u)$ and $L=L(x,u)$, then

$$\dot{H} = H_\lambda\dot{\lambda} + H_x\dot{x} + H_u\dot{u} \tag{11}$$

However, from Eqs. (7) and (8),

$$H_\lambda = f^T, \qquad H_x = -\dot{\lambda}^T \tag{12}$$

Using Eq. (12) in Eq. (11) along with Eq. (9), it follows that H is constant along an optimal trajectory. This fact, together with Eq. (10) for the case of t_f free, results in a first integral of the motion

$$H[x(t),\lambda(t),u(t)] = 0 \tag{13}$$

along the optimal path.

It is apparent that, for a given optimal control formulation, the optimal control depends only on $x(t_0)$. For implementation, we desire a control in feedback form

$$u(t) = u[x(t)] \tag{14}$$

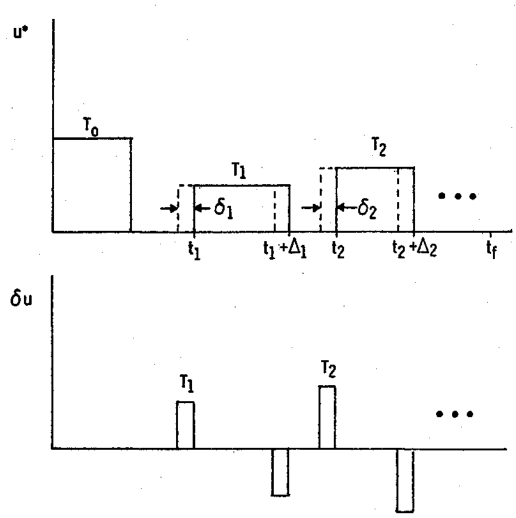

Fig. 1 Control variation due to variation in pulse initiation times.

This implies that the optimal control problem is solved repeatedly as the trajectory evolves, treating the current state as a new initial state, or we find optimal control problem formulations for which analytical solutions are possible. In nonlinear problems this usually can be done only by reducing the order of the model in Eq. (1). Thus, some degree of modeling simplification is usually present in optimal control design.

Optimal Pulse Control

In the case of pulse motor control, the thrust magnitude and duration for each pulse are fixed by the motor design. The only variable subject to control is the pulse initiation. In this case, the control is scalar ($m=1$) and $u^*(t)$ is defined by a set of optimal pulse initiation times. The variation $\delta u(t)$ arises from small variations in the pulse initiation times, as illustrated in Fig. 1. Note that the variation is not arbitrary, but appears as small-duration pulses at the start and end of each pulse in $u(t)$. The expression in Eq. (6) is still valid; however, Eq. (9) no longer holds since $\delta u(t)$ is no longer arbitrary.

In the following, it is assumed that H is a linear function of u. This amounts to assuming the following forms:

$$f(x,u) = f_0(x) + f_1(x)u \tag{15}$$

$$L(x,u) = L_0(x) + L_1(x)u \tag{16}$$

This assumption avoids discontinuities in H_u at the start and end of each pulse. Letting t_i denote the optimal pulsing time for the ith pulse, T_i the pulse magnitude, Δ_i the pulse width, and δ_i the variation in pulsing time, then to first order in δ_i we have

$$\delta J = \sum_{i=1}^{p} \delta_i T_i[H_u(t_i) - H_u(t_i+\Delta_i)] \leq 0 \tag{17}$$

where p is the number of pulses to be optimized. It is assumed in Fig. 1 that the first pulse is constrained to fire at t_0, although this may not be the case. Since the inequality in Eq. (17) must be met for arbitrary δ_i, it follows that a necessary condition for optimality of t_i is

$$H_u(t_i) = H_u(t_i+\Delta_i) \tag{18}$$

where

$$H_u(t_i) = H_u[x(t_i),\lambda(t_i)] \tag{19}$$

If the pulses are modeled as impulses of infinitesimal width, then Eq. (19) reduces to the condition

$$\frac{\mathrm{d}}{\mathrm{d}t}H_u(t_i) = 0 \tag{20}$$

which is the same condition derived in Ref. 3 for the case of impulsive thrust.

A pulse-control counterpart to condition (13) can also be derived. Define

$$H_0 = H[x(t),\lambda(t),0] \tag{21}$$

which is simply the Hamiltonian evaluated for $u(t)=0$. Now condition (13) is no longer valid since it relies on Eq. (9), which is now replaced by Eq. (19). However, Eq. (10) is still valid for a free terminal time problem. Hence, if $u(t_f)=0$ then

$$H_0(t_f) = 0 \tag{22}$$

This corresponds to the case where the trajectory is ended with a coasting arc. The question is: Does H_0 enjoy a property that in any way approximates Eq. (13)? Clearly

$$\dot{H}_0 = H_x(u=0)\dot{x} + H_\lambda(u=0)\dot{\lambda} = 0 \tag{23}$$

during coasting periods, which follows directly from Eqs. (11) and (12).

To relate H_0 from the ith coasting period to the $(i+1)$ coasting period, integrate H over the ith pulse, where

$$\dot{H} = \dot{H}_u \dot{u} \tag{24}$$

and $\dot{u}$ is a train of impulses

$$\dot{u}(t) = \sum_{i=1}^{p} T_i[\delta(t-t_i) - \delta(t-t_i-\Delta_i)] \tag{25}$$

Integrating Eq. (24) from $(t_i-\epsilon_1)$ to $(t_i+\Delta_i+\epsilon_2)$, where ϵ_1 and ϵ_2 are chosen to place the limits of integration anywhere in the coasting segments bordering the ith pulse, we obtain

$$H(t_i+\Delta_i+\epsilon_2) = H(t_i-\epsilon_1) + T_i[H_u(t_i) - H_u(t_i+\Delta_i)] \tag{26}$$

However, from Eq. (18) the second term on the right-hand side of Eq. (26) is zero. Therefore, it follows from Eqs. (10) and (26) that

$$H(t_i+\Delta_i+\epsilon_2) = H(t_i-\epsilon_1) = 0 \tag{27}$$

Now $H_0 = H$ during coasting arcs. At $\epsilon_1 = \epsilon_2 = 0$, H is discontinuous, but H_0 is not. Therefore, it follows that

$$H_0(t_i) = H_0(t_i+\Delta_i) = 0 \tag{28}$$

is a necessary condition for optimality when the trajectory ends in a coasting arc. This condition replaces Eq. (13) of the preceding section.

Optimal Pulse Motor Control Algorithm

In this section, a derivation is given for a pulse-control algorithm useful in pulse solid rocket applications. The derivation uses the two new necessary conditions for optimality, which are developed in the preceding section.

The equations of motion used to model the missile dynamics are

$$\dot{x} = V \tag{29}$$

$$\dot{E} = (T-D)V/W, \qquad E = z + V^2/2g \tag{30}$$

$$\dot{m} = -cT \tag{31}$$

where x is the horizontal range to the target, z the altitude, E the total missile energy (kinetic + potential) per unit weight (W), T the thrust, D the drag, V the velocity, and m the mass. It is assumed here that the missile flies at a constant altitude. The cost function is

$$J = V(t_f) \tag{32}$$

Hence, the objective is to find the pulsing times that result in maximum terminal velocity with terminal constraint

$$x(t_f) = x_f \tag{33}$$

used as a stopping condition.

The necessary conditions for optimality are

$$H_0 = \lambda_x V - \lambda_E DV/W = 0 \tag{34}$$

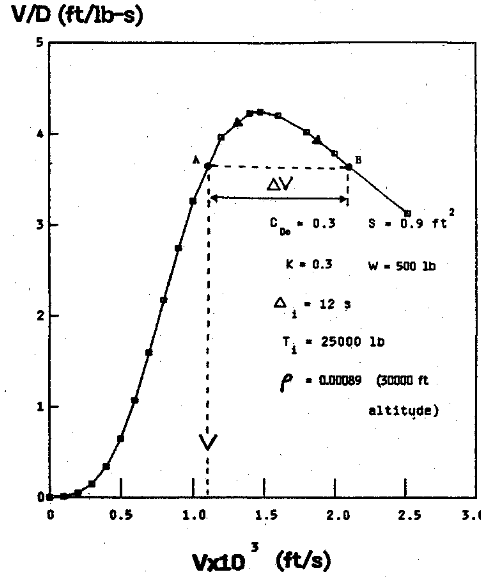

Fig. 2 Typical variation in V/D with velocity.

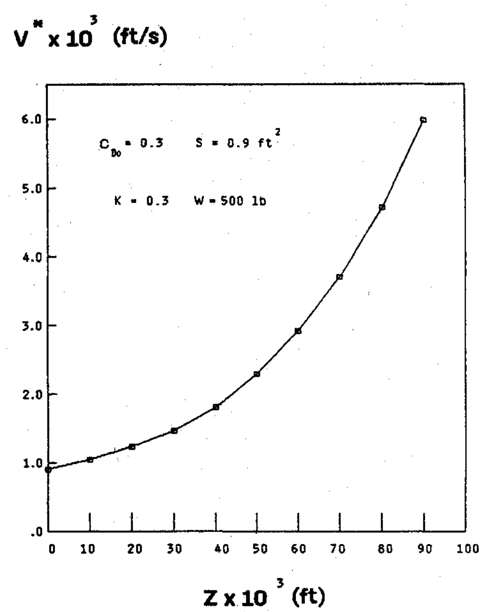

Fig. 3 Typical variation in max (V/D) velocity point with altitude.

at t_i and $t_i+\Delta_i$, and

$$H_T(t_i) = H_T(t_i+\Delta_i) \tag{35}$$

where, in this case,

$$H_T = \lambda_E V/W - c\lambda_m \tag{36}$$

Using Eq. (34) to eliminate λ_E in Eq. (36), then condition (35) can be expressed as

$$[\lambda_x V/D - c\lambda_m]_{t_i} = [\lambda_x V/D - c\lambda_m]_{t_i+\Delta_i} \tag{37}$$

The costate variables in Eq. (37) satisfy

$$\dot{\lambda}_x = H_x, \qquad \lambda_x(t_f) = \text{free} \tag{38}$$

$$\dot{\lambda}_m = H_m, \qquad \lambda_m(t_f) = 0 \tag{39}$$

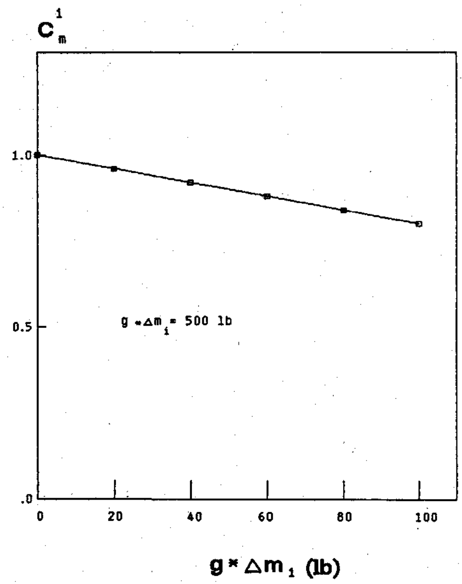

Fig. 4 Variation in the mass correction factor with propellant mass in the ith pulse.

where

$$H = \lambda_x V + \lambda_E (T - D) V/W - \lambda_m cT \tag{40}$$

The fact that λ_x is free at t_f is a result of the stopping condition in Eq. (33).

Next, we will examine the solution of these necessary conditions for a simplified case that neglects mass dynamics. The solution is then corrected to include the effects of these dynamics.

Constant Mass-Constant Altitude

In this case, the order of the system is reduced to 2 since $\dot{m} = 0$. Condition (37) becomes

$$[V/D]_{t_i} = [V/D]_{t_i + \Delta_i} \tag{41}$$

which is used as the pulse firing condition. Note that Eq. (41) does not require an analytical model for drag. However, this condition can be interpreted by examining the behavior of V/D vs V for a parabolic drag polar model. Let D have the form

$$D = qSC_{D0} + KL^2/qs \tag{42}$$

where

$$q = \rho V^2/2 \tag{43}$$

L is the lift and ρ the air density. For constant-altitude flight, $L = W$. Ignoring the variation of C_{D0} and K with Mach number, Eq. (42) can be written as

$$D = k_1 V^2 + k_2/V^2 \tag{44}$$

where

$$k_1 = \rho SC_{D0}/2, \qquad k_2 = 2KW^2/\rho S \tag{45}$$

Thus, V/D has the form

$$V/D = V^3/(k_1 V^4 + k_2) \tag{46}$$

For $k_1 > 0$ and $k_2 > 0$, this function has a single maximum at

$$V^* = (3k_2/k_1)^{1/4} \tag{47}$$

A typical graph of V/D is illustrated in Fig. 2.

During a pulse V increases monotonically, and during a coast it decreases monotonically. Let ΔV_i be the velocity increment due to the ith pulse. Then condition (41) states that the ith pulse should be fired when $V = V_i$, such that

$$V_i/D(V_i) = (V_i + \Delta V_i)/D(V_i + \Delta V_i) \tag{48}$$

This corresponds to moving from point A to point B on Fig. 1. Note that V_i and ΔV_i depend upon the magnitude and duration of the ith pulse. This is followed by a coasting arc, during which we move in the reverse direction until the firing condition for the next pulse is encountered. This is repeated for all $i = 1,\ldots,p$. Thus, the effect of the pulsing condition is (loosely speaking) to maximize the average value of V/D. The ΔV_i illustrated in Fig. 1 is for a total impulse of 30,000 lb-s, where ΔV_i is approximated by

$$\Delta V_i = T_i \Delta_i / m \tag{49}$$

Figure 3 illustrates the effect that altitude has on the pulse firing condition by showing the variation of V^* in Eq. (47) for the same parameter values in Fig. 2. In general, the dependence of C_{D0} and K on Mach number does not change the character of the solution described above, but it can change the optimal pulsing velocity significantly.

To implement the pulse-control law it is necessary to predict $V(t_i + \Delta_i)$ given $V(t_i)$ as if the pulse has been fired. An example would be to simply use

$$V(t_i + \Delta_i) = V(t_i) + T_i \Delta_i / m \tag{50}$$

which would be valid for a high T_i/D ratio. A pulse is then fired whenever the condition

$$[V/D]_{t_i} < [V/D]_{t_i + \Delta_i} \tag{51}$$

is encountered along the trajectory. In addition, the following conditions can also be used to trigger a pulse:

1) $V < V_{\min}$

2) $\alpha > \alpha_{\max}$

3) $T_{go} < \sum_{j=i}^{p} \Delta_j$

Conditions 1 and 2 ensure that minimum velocity and maximum angle-of-attack limits are not violated, and condition 3 ensures that all of the remaining pulses are utilized prior to intercept. Note that condition (51) may require that a pulse be fired immediately following the pulse used to launch the missile, in which case the equality in Eq. (48) may not be met. This simply means that the total impulse for the launching pulse should have been larger for the launch condition of interest. The launching pulse should be sized to yield a velocity at burnout that exceeds V_1, the pulse firing velocity for the subsequent pulse.

Variable Mass-Constant Altitude

The preceding section presented an exact analytical solution to the optimal pulsing problem for a constant-mass missile flying at constant altitude. In this section, an approximate correction factor to account for mass variations is derived. For the variable mass-constant altitude problem, the pulse firing condition in Eq. (37) must be used. To use this condition, $\lambda_m(t_i + \Delta_i)$ must be related to $\lambda_m(t_i)$. From Eq. (39),

$$\dot{\lambda}_m = \lambda_E (T - D) V/gm^2 \tag{52}$$

hence,

$$\frac{d\lambda_m}{dm} = \frac{\dot{\lambda}_m}{\dot{m}} = \frac{-k(t)}{m} \tag{53}$$

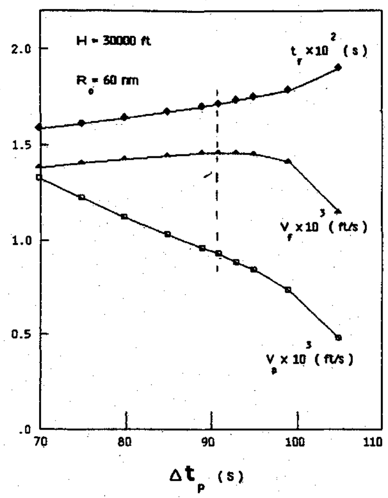

Fig. 5 Sensitivity results for constant altitude.

where

$$k(t) = (T-D)\lambda_E V/cTW \tag{54}$$

In order to integrate Eq. (53), approximate $k(t)$ by its average value

$$d\lambda_m = -\bar{k}\frac{dm}{m} \tag{55}$$

where

$$\bar{k} = [k(t_i) + k(t_i+\Delta_i)]/2 \tag{56}$$

$$k(t_i) = [\lambda_x (T_i - D)/cT_i]_{t_i}(V/D)_{t_i} \tag{57}$$

In Eq. (57) $\lambda_E V/W$ has been eliminated by using Eq. (34).

Integrating Eq. (55) from t_i to $t_i+\Delta_i$ establishes the following relation:

$$\lambda_m(t_i+\Delta_i) = \bar{k}\ell n[m_i/(m_i-\Delta m_i)] + \lambda_m(t_i) \tag{58}$$

where $m_i = m(t_i)$ and Δm_i is the propellant mass for the ith pulse. Using Eqs. (56-58) in Eq. (37) results in the following pulsing condition:

$$[V/D]_{t_i} = C_m^i * [V/D]_{t_i+\Delta_i} \tag{59}$$

where

$$C_m^i = (1-a_i^+\pounds_i)/(1+a_i\pounds_i)/(1+a_i\pounds_i) \tag{60}$$

$$a_i^+ = [(T_i - D)/2T_i]_{t_i+\Delta_i} \tag{61}$$

$$a_i = [(T_i - D)/2T_i]_{t_i} \tag{62}$$

$$\pounds_i = \ell n[m_i/(m_i - \Delta m_i)] \tag{63}$$

For $T_i \gg D$, C_m^i can be approximated by

$$C_m^i = (1-0.5\,\pounds_i)/(1+0.5\,\pounds_i) \tag{64}$$

Comparing Eq. (59) with Eq. (41), we can clearly see the effect of mass variation on the pulsing condition. Note that $C_m^i \leq 1.0$ and $C_m^i = 1.0$ when $\Delta m_i = 0$, which reduces Eq. (59) to the constant-mass pulsing condition. A graph of Eq. (64) is illustrated in Fig. 4.

Numerical Results

The pulse motor control algorithm was implemented in a simulation to evaluate optimality. This was done by using the constant-altitude-variable-mass solution, where the only approximation is embodied in Eq. (55). The study was conducted on a two-pulse motor, where the pulsing time for the second pulse was the only variable. The drag coefficient was tabulated as a function of Mach number, angle of attack, and altitude.

The motor parameters used in this study were defined as follows:

$$T_0 = 2500 \text{ lb}, \qquad T_1 = 2500, \qquad \Delta_0 = 36.4 \text{ s}$$

$$\Delta m_0 = 11.0 \text{ slugs}, \qquad \Delta m_1 = 3.8, \qquad \Delta_1 = 12.2$$

$$c = 0.000125 \text{ slugs/lb-s} \tag{65}$$

The starting weight for the missile was 870 lb. All runs were made for an altitude of 30,000 ft and an initial velocity of 613.8 ft/s in level flight. In addition to letting the pulse motor control algorithm fire the pulse, the pulse firing time was varied manually to determine the sensitivity of terminal velocity V_f and final time t_f, and to assess optimality of the algorithm.

Figure 5 summarizes the results for a constant-altitude trajectory to a range of 60 nm by showing the variation of V_f, t_f, and the pulsing velocity V_p as a function the time delay between pulses Δt_p. The algorithm fired the pulse at $\Delta t_p = 91$ s. The pulsing velocity was 917 ft/s, and the final velocity was 1453 ft/s. Note that despite the fact that V_f is nearly a flat function of Δt_p in this region, the algorithm was successful in finding the optimal pulse firing time. This dramatically demonstrates the validity of the pulse firing condition in Eq. (59), and that the approximate correction for mass variation is, for all practical purposes, exact. This experiment was repeated for several altitudes, and the algorithm always found a precise optimal firing time. It should be noted that, although V_f is insensitive to pulse firing time in the vicinity of the optimal, this does not imply that performance is not improved by a two-pulse motor. For small values of pulse delay time (not illustrated) the missile was unable to reach the target range.

Summary

Two new necessary conditions for optimality are derived that should prove useful in analyzing the problem of optimal pulse control. The derivation is based on the calculus of variations. The validity and usefulness of these conditions are demonstrated on a missile application. The main result is a simple pulse firing condition based on the ratio of missile speed to aerodynamic drag. It is shown that the effect tends to maximize the average of this ratio over the flight time.

Acknowledgments

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